Minkowski dimension: various definitions, including Whitney cubes

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The first, and easiest to define (and compute), definition of the dimension.

Convenient notation: $|E| = \operatorname{diam} E$. We'll use it when not confusing.

Def. Let K be a totally bounded ($\forall \varepsilon > 0\ \exists$ finite ε-net) subset of a metric space (X, ρ).

$$N(\varepsilon, K) := \min \{ n : K \subset \bigcup_{j=1}^{n} K_j,\ \operatorname{diam} K_j =: |K_j| < \varepsilon \}.$$

(Upper/lower) Minkowski dimension of K

$$\overline{Mdim}\ (\underline{Mdim})\, K = \overline{\lim_{\varepsilon \to 0}}\ (\underline{\lim_{\varepsilon \to 0}})\ \frac{\log N(\varepsilon, K)}{\log \frac{1}{\varepsilon}}.$$

If lim exists: $Mdim\, K$, Minkowski dimension of K.

$N(\varepsilon, K)$ good for upper bounds. For lower bounds, & $P(\varepsilon, K) := \max \{ n : \exists x_1, \dots, x_n : \rho(x_j, x_k) \geq \varepsilon \}$.

$$P(2\varepsilon, K) \leq N(\varepsilon, K) \leq P(\tfrac{\varepsilon}{2}, K)$$

(each K_j contains at most one x_j) ($\because$ covering by $B(x_j, \frac{\varepsilon}{2})$).

Thus
$$\overline{Mdim}\ (\underline{Mdim}) = \overline{\lim_{\varepsilon \to 0}}\ (\underline{\lim_{\varepsilon \to 0}})\ \frac{\log P(\varepsilon, K)}{\log \frac{1}{\varepsilon}}.$$

Examples:

1) $[0,1]$ has $Mdim = 1$

2) $[0,1]^d$ has $Mdim = d$. In general, open, bounded in $\mathbb{R}^d$ has $Mdim = d$.

3) $f : (X, \rho_X) \to (Y, \rho_Y)$ – Bilipschitz ($\rho_X(x_1, x_2) \lesssim \rho_Y(f(x_1), f(x_2)) \lesssim \rho_X(x_1, x_2)$), then $\overline{Mdim}\, fK = \overline{Mdim}\, K$, same for $\underline{Mdim}$.

Pf. $N(C_2 \varepsilon, fK) \leq N(\varepsilon, K)$, $N(\frac{\varepsilon}{C_1}, K) \leq N(\varepsilon, fK)$.

4) C – standard Cantor set.
$3^{-n} \leq \varepsilon < 3^{-n+1} \Rightarrow N(\varepsilon, C) \leq 2^n$, $P(\varepsilon, C) \geq 2^{n-1}$. Thus
$$Mdim\, C = \lim_{n \to \infty} \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3}.$$

5) Let us generalize. Let $2\ell_n < \ell_{n-1}$, $\ell_0 = 1$.
Let $C_0 = [0,1]^d$, C_1 – take 2^d corner cubes of size ℓ_1 (and diam $\sqrt{d}\,\ell_1$),
C_2 – divide each cube from C_1 into 2^d cubes of size ℓ_2 ...
C_n – consists of 2^{nd} cubes of size ℓ_n.
$C := \bigcap C_n$.

Then, for $\ell_n \leq \varepsilon < \ell_{n-1}$, $N(\sqrt{d}\,\varepsilon, K) \leq 2^{nd}$ (cover by C_n), $P(\sqrt{d}\,\varepsilon, K) \geq 2^{(n-1)d}$ (at most one in C_{n-1}),

so
$$\overline{Mdim}\, C = \overline{\lim}\ \frac{\log N(\sqrt{d}\,\varepsilon, K)}{\log \frac{1}{\varepsilon}} = \overline{\lim}\ \frac{nd \log 2}{\log \frac{1}{\ell_n}} = \frac{d}{\underline{\lim}\, |\log_2 \sqrt[n]{\ell_n}|}.$$

Same with $\underline{Mdim}\, C = \dfrac{d}{\overline{\lim}\, |\log_2 \sqrt[n]{\ell_n}|}$.

BTW, for, say,
$$\frac{\ell_n}{\ell_{n+1}} = \begin{cases} 3, & 2^{2k} \leq n < 2^{2k+1} \\ 5, & 2^{2k+1} \leq n < 2^{2k+2} \end{cases}$$
we have $\overline{Mdim} > \underline{Mdim}$.

6) (Disappointing). $K = \{0\} \cup \{\frac{1}{n},\ n \in \mathbb{N}\}$.

$N^{-2} \geq \varepsilon > (N+1)^{-2} \Rightarrow \left(n \leq N \Rightarrow \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > \frac{1}{n^2} \geq \frac{1}{N^2} \geq \varepsilon \right) \Rightarrow P(\varepsilon, K) \geq \sqrt{\frac{1}{\varepsilon}} - 1$.

But $N(\varepsilon, K) \leq N + 1 + N \leq \frac{2}{\sqrt{\varepsilon}} + 1$ (since $[0, \frac{1}{N+1}]$ can be covered by $N+1$ intervals of size $\frac{1}{(N+1)^2}$).

Thus $Mdim\, K = \frac{1}{2}$.

Countable set with positive $Mdim$! Disappointing!

Other definitions for $\mathbb{R}^d$:

1) Let *b-adic interval* be an interval of the form $[kb^{-n}, (k+1)b^{-n})$, b-adic cube be a product in $\mathbb{R}^d$ of d intervals of the same size.
Reason to use: hierarchical structure: each n-level b-adic cube has b^d non-intersecting children of level $(n+1)$.
$N_n^b(K) := \#\{b^{-n} \text{ cubes } Q : Q \cap K \neq \emptyset\}$.
If $b^{-n} \le \varepsilon < b^{-n+1}$, then $N(\sqrt{d}\varepsilon, K) \le N_n^b(K)$, and
On the other hand, for each set E with $\operatorname{diam} E \le \varepsilon$, we have $E \subset B(x,\varepsilon)$ for any $x \in E$. Moreover, if Q is b^{-n} cube, $Q \cap E \neq \emptyset \Rightarrow Q \subset B(x, \varepsilon + \sqrt{d}b^{-n}) \subset B(x,(1+\sqrt{d})\varepsilon)$.
Thus $N_n(E)\,\varepsilon^d \le \operatorname{Vol}(B(x,(1+\sqrt{d})\varepsilon)) = \sigma_d \varepsilon^d (1+\sqrt{d})^d$, so
$N_n^b(E) \le \sigma_d (1+\sqrt{d})^d =: C_d$.
so $N(\varepsilon, K) \le C_d N_n^b(K)$. Thus
$\overline{\operatorname{Mdim}}\ (\underline{\operatorname{Mdim}} K) = \overline{\lim_{n\to\infty}}\ (\underline{\lim_{n\to\infty}})\ \frac{\log N_n^b(K)}{n \log b} = \frac{1}{n}\log_b N_n^b(K)$.

Example. $S \subset \mathbb{N}$, $A_S = \{x = \sum_{k\in S} x_k 2^{-k}, x_k \in \{0,1\}\}$.
For $b=2$: $N_n^2(S) = 2^{\#\{S \cap \{1,\dots,n\}\}}$. Thus
$\overline{\operatorname{Mdim}}\, A_S = \overline{\lim}\ \frac{\#\{S \cap \{1,\dots,n\}\}}{n}$ – upper
$\underline{\operatorname{Mdim}}\, A_S = \underline{\lim}\ \dots$ – lower density of S.

2) For $K \subset \mathbb{R}^d$, let $K_\varepsilon := \{x : \operatorname{dist}(x,K) < \varepsilon\}$.
Note that
- $\operatorname{Vol} K_\varepsilon \le N(\varepsilon, K)\operatorname{Vol}(B_{2\varepsilon}) \le N(\varepsilon,K)\,\sigma_d \varepsilon^d 2^d$.
 (since $K \subset \bigcup K_j$, $K^\varepsilon \subset \bigcup K_j^\varepsilon$, $K_j \subset B(x_j,\varepsilon) \Rightarrow K_j^\varepsilon \subset B(x_j, 2\varepsilon)$).
- $\operatorname{Vol} K_\varepsilon \geq P(\varepsilon, K)\operatorname{Vol}(B_{\varepsilon/2}) = P(K,\varepsilon)\,\sigma_d \varepsilon^d 2^{-d}$.
 (since for ε-net $(x_j)_{j=1}^{P(\varepsilon,K)}$, $B(x_j, \frac{\varepsilon}{2}) \subset K^\varepsilon$, and they do not intersect).
so $\overline{\operatorname{Mdim}}\, K = \overline{\lim}\ \frac{\log \operatorname{Vol} K_\varepsilon}{\log \frac{1}{\varepsilon}} + d$, same for $\underline{\operatorname{Mdim}}$.

3) Using Whitney decomposition.
Let U be an open set. $\forall x \in U$ $\exists$ maximal dyadic Q_x, such that $Q_x \subset U$, and $|Q_x| \le \operatorname{dist}(Q_x, \partial U)$.
If $x \neq y$, then either $Q_x = Q_y$ or $Q_x \cap Q_y = \emptyset$.
Thus $U = \bigsqcup Q_x$ – disjoint union.
Also, if Q' is parent of Q_x, then $|Q'| > \operatorname{dist}(Q', \partial U)$.
Thus $2|Q_x| > \operatorname{dist}(Q', \partial U)$, and $\operatorname{dist}(Q_x, \partial U) \le |Q_x| + \operatorname{dist}(Q', \partial U) \le 3|Q_x|$.
Thus, for our Whitney decomposition, we have $|Q_x| \le \operatorname{dist}(Q_x, \partial U) \le 3\,|Q_x|$.
Remark. Any decomposition with
$\frac{1}{\lambda}\operatorname{dist}(Q, \partial U) \le |Q| \le \lambda \operatorname{dist}(Q, \partial U)$ is called Whitney decomposition. We just gave one example.

Def. Let $K \subset \mathbb{R}^d$ be a compact, $\{Q\}$ be a Whitney decomposition
$\alpha := \alpha(K) = \inf\{\alpha : \sum_{Q:|Q|\le 1} |Q|^\alpha < \infty\}$.
any

To show that α does not depend on W – the Whitney decomposition – let us relate it to the volume of a nbhd.
Let $C_n := \#\{Q \in W : 2^{-n-1} < |Q| \le 2^{-n}\}$.
Then $\sum |Q|^\alpha \asymp \sum 2^{-n\alpha} C_n$. (up to a factor of 2^α).
Let $D_n := \operatorname{Vol}\{K^{2^{-n-1}} \setminus K^{2^{-n}}\}$.
Observe that for some ℓ depending on λ,
if $Q \cap K^{2^{-n+1}} \setminus K^{2^{-n}} \neq \emptyset \Rightarrow 2^{-n-\ell} \le |Q| \le 2^{-n+\ell}$.
Thus $C_n \le 2^{(n+\ell)d}\left(\sum_{i=n-\ell}^{n+\ell} D_i\right)$ and
$D_n \le (2^{-n+\ell})^d \sum_{i=n-\ell}^{n+\ell} C_i$. Thus
$\sum 2^{-n\alpha} C_n < \infty \iff \sum 2^{nd-n\alpha} D_n < \infty$.
Now we can see
Lemma. For any $K \subset \mathbb{R}^d$ - compact, $\alpha \le \overline{\operatorname{Mdim}}\, K$.
$\operatorname{Vol} K = 0 \Rightarrow \alpha = \overline{\operatorname{Mdim}}\, K$

$\mathrm{Vol}\,k = 0 \Rightarrow \quad \alpha = \overline{\mathrm{Mdim}}\,k$

Pf. If $\beta > \overline{\mathrm{Mdim}}\,k$, and $\beta > \beta_1 > \overline{\mathrm{Mdim}}\,k$

$\sum_{i=n}^{\infty} D_i \le \mathrm{Vol}\,k^{2^{-n+1}} \le 2^{n(\beta_1-d)}$ for large n.

Then $D_n \le 2^{n(\beta_1-d)}$

so $\sum 2^{nd-n\beta} D_n \le \sum 2^{nd-n\beta}\, 2^{n(\beta_1-d)} < \infty$.

If $\mathrm{Vol}\,k = 0$ then

$\mathrm{Vol}\,k^{2^{-n+1}} = \sum_{i=n}^{\infty} D_i$.

Note that if $\sum 2^{nd-n\alpha} D_n < \infty$, then

$2^{nd-n\alpha} D_n \to 0$, and, in particular, $D_n \le C\, 2^{n\alpha-nd}$.

Then $\mathrm{Vol}\,k^{2^{-n+1}} \le C \sum_{i=n} 2^{i(\alpha-d)} \le C\, 2^{n(\alpha-d)}$ for all n. Thus

$\overline{\mathrm{Mdim}} \le \alpha$